COMPLETE TOPOLOGIES ON SPACES OF BAIRE MEASURE

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ABSTRACT. Let X be a completely regular Hausdorff space, let L be the linear space of all finite linear combinations of the point measures on X and let M_σ denote the space of Baire measures on X. The following is proved: *If M_σ is endowed with the topology of uniform convergence on the uniformly bounded, equicontinuous subsets of $C^b(X)$, then M_σ is a complete locally convex space in which L is dense and whose dual is $C^b(X)$*, provided there are no measurable cardinals. A complete description of the situation in the presence of measurable cardinals is also given. Let M_c be the subspace of M_σ consisting of those measures which have compact support in the realcompactification of X. The following result is proved: *If M_c is endowed with the topology of uniform convergence on the pointwise bounded and equicontinuous subsets of $C(X)$, then M_c is a complete locally convex space in which L is dense and whose dual is $C(X)$*, provided there are no measurable cardinals. Again the situation if measurable cardinals exist is described completely. Let M denote the Banach dual of $C^b(X)$. The following is proved: *If M is endowed with the topology of uniform convergence on the norm compact subsets of $C^b(X)$, then M is a complete locally convex space in chich L is dense*. It is also proved that M_σ is metrizable if and only if X is discrete and that the metrizability of either M_c or M is equivalent to X being finite. Finally the following is proved: *If M_c has the Mackey topology for the pair $(M_c, C(X))$, then M_c is complete and L is dense in M_c*.

Introduction. Throughout the paper, X will denote a completely regular Hausdorff space (unless a statement is made to the contrary), C or $C(X)$ will denote the linear space of all continuous real-valued functions on X and C^b or $C^b(X)$ will denote the subspace of C consisting of those functions which are uniformly bounded. The linear space of all real-valued functions on X which are zero on the complement of a finite subset will be denoted by L or $L(X)$. (Hence L is a realization of the free linear space generated by X over the reals.) The space L should be conceived as the space of all finite linear combinations of the point measures of unit mass on X. The spaces L and C may be paired in

the sense of Bourbaki by the bilinear form defined by $\langle \xi, f \rangle = \Sigma\{\xi(x)f(x): x \in X\}$ for all $\xi \in L$ and $f \in C$. The same bilinear form defines by restriction a pairing of L with C^b. Generally, the purpose of the paper will be to investigate the completion of L with respect to several natural locally convex topologies on L compatible with the pairs (L, C) and (L, C^b). The method used is to exploit systematically the well-known theorem of A. Grothendieck concerning complete locally convex spaces in the light of various results taken from the theory of measures in topological spaces. The net result will be to find complete spaces of Baire measures in which L is a dense subspace.

The origin of the problem occurs in M. Katětov's papers [10] and [11]. Katětov's result can be described as follows. Identify X with a subset of L by the map $x \rightsquigarrow \delta_x$, where δ_x denotes the characteristic function of $\{x\}$. (Make this identification, and assume for the remainder of the paper that X is a subset of L.) Katětov considers a topology e on L which is the finest locally convex topology on L whose restriction to X is the original topology of X. He then proves that if X is compact, then the completion of (L, e) is the Banach dual of C. In [22] V. Pták extends this result to the case that X is pseudocompact (Theorem 7, p. 187). In [23] D. A. Raikov characterizes the topology e on L as the topology of uniform convergence on the pointwise bounded, equicontinuous subsets of C for the pair (L, C). Since it is well known that if X is pseudocompact, then the Banach dual of C is the space of all finite signed Baire measures on X, the result of Katětov-Pták may be stated as follows:

Theorem. *Let X be pseudocompact. Then the completion of L for the topology of uniform convergence on the pointwise bounded, equicontinuous subsets of C is the space of all finite signed Baire measures on X.*

The main results of the present paper are essentially generalizations of the above theorem to arbitrary completely regular spaces. (See Theorems 8.7, 8.15, 9.5, and 10.6.) Since if X is not pseudocompact, then $C \ne C^b$, it is clear that more than one generalization is conceivable.

The paper itself has three major divisions. In part I, *Baire measure on topological spaces*, there are four sections. The purpose of this part is to establish the necessary background from the theory of measures in topological spaces. There are relatively few proofs, and the reader is referred for further details to the following : [8], [12], [13], [18], [19] and [25].

In part II, *Topologies on the space L*, there are three sections. In §5 there are some preliminary results concerning equicontinuous families. In §6 the topology e is considered, and the improtant properties of this topology (obtained by Raikov in [23]) are established. In §7 a topology e^b analogous to e is

considered. Properties similar to those of e are established for e^b. In particular, it is shown that e^b is the finest locally convex topology on L for which X is a bounded subset and whose restriction to X is the original topology on X. Also in this section the topology s on L of uniform convergence on the norm compact sets in C^b is investigated. A categorical property of this topology is established. This is that s is the finest locally convex topology on L for which X is a totally bounded subset and whose restriction to X is the original topology on X.

Part III, *Completions of L*, contains three sections. The main results of the paper are contained here. In §8, it is shown that (except for some assumptions about measurable cardinals), the completion of (L, e^b) is the space of all Baire measures on X. In §9 it is shown that the completion of (L, e) is the space of all Baire measures on X which have compact support in the realcompactification of X. Also the completion of (L, s) is shown to be the Banach dual of C^b. Finally in §10 the Mackey topology for the pair (L, C) is considered. The completion of L with this topology is shown to be the space of all Baire measures with compact support if X is realcompact.

I. BAIRE MEASURES IN TOPOLOGICAL SPACES

In order to make the paper self-contained, some facts from the theory of measures in topological spaces will be reviewed here. The reader is referred to the works of Katětov [8]; Kirk [12], [13]; Knowles [16]; Moran [18], [19]; and Varadarajan [25].

1. The Banach dual of C^b. Let $\mathcal{Z}$ denote the family of zero sets in X and let $\mathcal{F}$ denote the algebra of subsets of X generated by $\mathcal{Z}$. The *Baire algebra* of X is the σ-algebra generated by $\mathcal{Z}$, or, equivalently, the smallest σ-algebra with respect to which the functions in C (or C^b) are measurable. The *Borel algebra* of X is the σ-algebra generated by the closed sets in X. The Borel algebra always contains the Baire algebra, but in general they are not equal.

Definition 1.1. A finitely-additive real-valued set function m on $\mathcal{F}$ is *regular* if for each set $W \in \mathcal{F}$ and for each positive number ϵ, there is a $Z \in \mathcal{Z}$ such that $Z \subset W$ and such that $|m(W')| \leq \epsilon$ whenever $W' \in \mathcal{F}$ and $W' \subset W - Z$.

The definitions of a regular set function on the Baire algebra and on the Borel algebra are essentially the same, except that in the case of the Borel algebra, one replaces $\mathcal{Z}$ with the family of closed sets in X. Let M denote the space of all finitely-additive, real-valued regular set functions m on $\mathcal{F}$ with the property that $\sup\{|m(W)|: W \in \mathcal{F}\} < \infty$. It can be shown (see [14]) that, with the usual pointwise definitions of addition, scalar multiplication and order, M is an L-space in the sense of Kakutani. The following is due to A. D. Alexandrov [1]. (See [14] for related results.)

Theorem 1.2. *The Banach dual of C^b is isomorphic as on L-space to M. The isomorphism may be realized by $\phi \leftrightarrow m$ where $\phi(f) = \int_X f\, dm$ for all $f \in C^b$. (The integral is the Riemann integral.)*

For each functional ϕ in the Banach dual of C^b (for each $m \in M$), denote by ϕ^+, ϕ^- and $|\phi|$ (by m^+, m^- and $|m|$) the positive, negative and total variation of ϕ (of m) respectively. The following result will be useful below. (See [14].)

Proposition 1.3. *Let $m \in M$. For every positive number ϵ, there are disjoint zero sets Z_1 and Z_2 such that $m^+(X - Z_1) \le \epsilon$ and $m^-(X - Z_2) \le \epsilon$.*

Definition 1.4. Let ϕ belong to the Banach dual of C^b. Then ϕ is *σ-additive* if for every decreasing sequence $\{f_n\} \subset C^b$ with $f_n \downarrow 0$ pointwise, it follows that $|\phi|(f_n) \to 0$. The set function $m \in M$ is *σ-additive* if for every decreasing sequence of sets $\{Z_n\} \subset \mathfrak{Z}$ with $\bigcap\{Z_n : n \in N\} = \varnothing$, it follows that $|m|(Z_n) \to 0$.

The following definition gives an important continuity condition which is stronger than that of σ-additivity.

Definition 1.5. Let ϕ belong to the Banach dual of C^b. Then ϕ is *net-additive* if for every net $\{f_i\} \subset C^b$ which is directed downward to zero, it follows that $|\phi|(f_i) \to 0$. (The net $\{f_i\}$ is *directed downward to zero* if given f_i and f_j, then there is f_k such that $f_k \le \inf(f_i, f_j)$, and if $\inf\{f_i(x)\} = 0$ for all $x \in X$.) The set function $m \in M$ is *net-additive* if for every net $\{Z_i\} \subset \mathfrak{Z}$ which is directed downward to the empty set, it follows that $|m|(Z_i) \to 0$. (The net $\{Z_i\}$ is *directed downward to the empty set* if given Z_i and Z_j, there is a Z_k such that $Z_k \subset Z_i \cap Z_j$ and if $\bigcap_i\{Z_i\} = \varnothing$.)

As might be expected, there is a relation between σ-additive and net-additive functionals and the corresponding types of set functions.

Proposition 1.6. *Let ϕ belong to the Banach dual of C^b and let $m \in M$ correspond to ϕ under the isomorphism of Theorem 1.2. Then the following hold:*
  (1) *ϕ is σ-additive if and only if m is σ-additive.*
  (2) *ϕ is net-additive if and only if m is net-additive.*

We will use M_σ and M_τ to denote the spaces of all σ-additive and all net-additive elements of M respectively. These are ideals (which are even Dedekind complete) in the L-space M. Hence if $m \in M_\sigma$ (M_τ), then m^+, m^-, $|m| \in M_\sigma$ (M_τ). It is easy to verify that if $m \in M_\sigma$, then m is a countably-additive set function on $\mathfrak{F}$. Hence by the Carathéodory process, every element in M_σ has a unique extension to a finite, regular signed Baire measure. For this reason we may identify M_σ with the space of all finite, signed Baire measures. (Such measures are regular of necessity.)

The definition of a net-additive set function on the Borel algebra of X is the

same as Definition 1.5, except that zero sets are replaced by closed sets. The following is proved in [13].

Proposition 1.8. *Let* $m \in M$ *be net-additive. Then there is a unique net-additive, regular signed Borel measure* μ *such that* $\mu(W) = m(W)$ *for all* $W \in \mathcal{F}$.

In view of Proposition 1.8, it is not difficult to verify that M_r may be identified with the space of all net-additive, regular signed Borel measures.

2. Supports.

Definition 2.1. Let $m \in M$. Then the *support of* m is defined by supp $m = \bigcap\{Z : Z \in \mathcal{Z}$ and $|m|(Z) = |m|(X)\}$. (Note that supp m is closed.)

It is easy to verify, since M_r is an ideal in M, that supp m is not empty whenever $0 < m \in M_r$. However, in general, the support of m may be empty even if $m \in M_\sigma$. (See [12].) A space X in which every element of M_σ has a nonempty support is called *measure compact* by Moran [18] and *B-compact* by Kirk [12]. It is shown by Pym in [21] that these are exactly the spaces in which $M_\sigma = M_r$. The following result is due to Katětov [8]. (See also Varadarajan [25].) First recall that a set Y has a *measurable cardinal* if there is a probability measure defined on the σ-algebra of all subsets of Y which is zero on all singleton sets. A set Y has a *nonmeasurable cardinal* if it does not have a measurable cardinal. It is consistent with the axioms of set theory to assume that all sets have nonmeasurable cardinals (although it is not known whether or not this assumption is independent). See [4, pp. 164–166].) Furthermore, if the continuum hypothesis holds, then the continuum has a nonmeasurable cardinal.

Proposition 2.2. *Let* X *be paracompact and assume that every closed, discrete subspace of* X *has a nonmeasurable cardinal. Then* X *is measure compact.*

The following useful result is also proved by Varadarajan in [25].

Proposition 2.3. *Let* X *be paracompact and let* $m \in M_r$. *Then* supp m *is a closed Lindelöf space.*

3. The spaces M_s.
Since we cannot guarantee that measurable cardinals do not exist, we must consider a subspace of M_σ which was introduced by Dudley in [3] and which will play an important role later. We first recall some definitions.

A *pseudometric* on X is a function d from $X \times X$ into the real numbers which satisfies all the axioms of a metric except that $d(x, y) = 0$ need not imply that $x = y$. The pseudometric d is *continuous* if it is a continuous function when $X \times X$ has the product topology, and it is *bounded* if there is a number K such that $d(x, y) \leq K$ for all $x, y \in X$. If d is a pseudometric on X, the relation $x \sim y$ if $d(x, y) = 0$ is an equivalence relation on X. Let X^* denote the set of

equivalence classes and for $[x]$, $[y] \in X^*$ define $d^*([x], [y]) = d(x, y)$. Then (X^*, d^*) is a metric space which is called the *metric space associated with the pseudometric d*.

Definition 3.1. Let d be a continuous pseudometric on X. An element $m \in M_\sigma$ is called *d-separable* if there is a d-closed set $Z \subset X$ such that $|m|(X - Z) = 0$ and such that Z contains a countable d-dense subset.

Note that if d is a continuous pseudometric on X and if $Z \subset X$ is d-closed, then $Z \in \mathcal{Z}$. Indeed if Z is not empty, then the function f defined by $f(x) = d(x, Z) = \inf\{d(x, y): y \in Z\}$ belongs to C since d is continuous, and Z is the zero set of f since Z is d-closed. (Of course, if $Z = \emptyset$, then the result is clear.)

Definition 3.2. By M_s denote the set of all those set functions $m \in M_\sigma$ such that m is d-separable for every continuous pseudometric d on X.

It is not hard to verify that $m \in M_s$ if and only if $m \in M_\sigma$ and m is d-separable for every bounded continuous pseudometric on X. It is clear that if $m \in M_s$, then m^+, m^-, $|m| \in M_s$ so that M_s is an ideal in M. The following notion is due to Granirer [8].

Definition 3.3. The space X is a *D-space* if whenever d is a continuous pseudometric on X, then every d-discrete subspace of X has a nonmeasurable cardinal.

The next result is essentially due to Marczewski and Sikorski [20], and it is also discussed by Dudley in [3].

Proposition 3.4. *The space X is a D-space if and only if $M_s = M_\sigma$.*

Since, as we have seen above, it is consistent to assume that all cardinals are nonmeasurable, it follows that it is consistent with the axioms of set theory to assume that $M_s = M_\sigma$ is valid for all spaces X. Since metric spaces are, in particular, paracompact, a proof of Proposition 3.4 may be based on Proposition 2.2, Proposition 2.3 and the following lemma (which will be useful later).

Lemma 3.5. *Let d be a continuous pseudometric on X and let $m \in M_\sigma$. Let $\Theta: X \to X^*$ be the canonical map of X into the associated metric space. Then $\Theta^{-1}[W]$ is a Baire set (a zero set) in X whenever W is a Baire set (a closed set) in X^*. Furthermore, if m_d is defined by $m_d(W) = m(\Theta^{-1}[W])$ for every Baire set W in X^*, then m_d is a regular, signed Baire measure on X^*.*

Proof. Since Θ is continuous, it is clear that $\Theta^{-1}[W]$ is a Baire set in X if W is a Baire set in X^*. That m_d is a regular, signed Baire measure on X^* can be verified in a straightforward manner. It can be observed that if W is closed in X^*, then $\Theta^{-1}[W]$ is a zero set in X since $\Theta^{-1}[W] = \{x \in X: f(x) = 0\}$ where $f(x) = \inf\{d(x, y): y \in \Theta^{-1}[W]\}$. The proof is complete.

4. **The order dual of** C. In this section we will consider a representation theorem for the order dual of C analogous to Theorem 1.2. Recall that a linear functional ϕ on the Riesz space C is *order bounded* if, for every $0 \leq f \in C$, $\sup\{\phi(g): g \in C \text{ and } 0 \leq g \leq f\} < \infty$. The set of all order bounded linear functionals on C forms a Dedekind complete Riesz space called the *order dual* of C. The following result is due to Hewitt [7]. (Also see [5].) We will give a proof based on Theorem 1.2 which is shorter than Hewitt's original proof. (The proof that ϕ is an integral given below is due W. A. J. Luxemburg.)

Proposition 4.1. *Let ϕ be a nonnegative linear functional on C. Then there is a regular Baire measure m on X such that $\phi(f) = \int_X f \, dm$ for all $f \in C$. Furthermore, if X is realcompact, then m has compact support.*

Proof. If ϕ is not an integral, then there is a decreasing sequence $\{f_n\} \subset C$ such that $f_n \downarrow 0$ pointwise on X while $\phi(f_n) \geq 2\alpha\phi(1) > 0$ for all n, where α is a fixed positive number. Define a function $f = \sum_{n=1}^{\infty} (f_n - \alpha)^+$. It is simple to verify that $f \in C$. However, for all n,

$$\phi(f) \geq \sum_{k=1}^{n} \phi((f_k - \alpha)^+) \geq \sum_{k=1}^{n} (\phi(f_k) - \alpha\phi(1)) \geq n\alpha\phi(1).$$

Hence $\phi(f) = \infty$ which is a contradiction.

Since ϕ is nonnegative, its restriction to C^b belongs to the Banach dual of C^b. Hence by Theorem 1.2, there is a nonnegative $m \in M$ such that $\phi(f) = \int_X f \, dm$ for all $f \in C^b$. Furthermore, m is a Baire measure by Proposition 1.6. If $0 \leq f \in C$, set $f_n = \inf(f, n)$. Then $\{f_n\} \subset C^b$ and $f_n \uparrow f$ pointwise on X. Hence by the monotone convergence theorem and the fact that ϕ is an integral, $\phi(f) = \lim \phi(f_n) = \lim \int_X f_n \, dm = \int_X f \, dm$. By the linearity of ϕ, it follows that $\phi(f) = \int_X f \, dm$ for all $f \in C$.

Now assume that X is realcompact. If $Y \subset X$, let $\overline{Y}$ denote the closure of Y in βX (the Stone-Čech compactification of X). The family of zero sets $\{Z \in \mathcal{Z}: m(Z) = m(X)\}$ is clearly closed under finite intersections. Hence the set $G = \bigcap\{\overline{Z}: Z \in \mathcal{Z} \text{ and } m(Z) = m(X)\}$ is a nonempty compact subset of βX. Of course, we assume that $m \neq 0$. The proof will be complete by Definition 2.1 if we can show that $G \subset X$.

If G is not a subset of X, let $x_0 \in G - X$. Since X is realcompact, there is a function $g \in C$ with the property that $\overline{g}(x_0) = +\infty$ (where $\overline{g}$ is the unique continuous function taking values in the extended real numbers obtained by extending g to βX). Define $Z_n = \{x \in X: g(x) \geq n\}$. Then for each n, Z_n is a nonempty zero set in X and $Z_{n+1} \subset Z_n$. Furthermore, $m(Z_n) > 0$ for all n. Indeed, if $m(Z_{n_0}) = 0$, let $Z = \{x \in X: g(x) \leq n_0\}$. Then $m(Z) = m(X)$ so that

$x_0 \in \overline{Z}$ by the definition of G. But this means $\overline{g}(x_0) \le n_0$ which is contrary to assumption.

Since $Z_n \downarrow \varnothing$, it follows that $m(Z_n) \downarrow 0$. Hence, by passing to a subsequence if necessary, we may assume that $m(Z_{n+1}) < m(Z_n)$ for all n. Let $\alpha_n = m(Z_n)$, and let $\{f_n\}$ be a sequence in C such that $0 \le f_n$, $f_n = (\alpha_{3n+1} - \alpha_{3n+2})^{-1}$ on the set $W_n = \{x \in X: 3n+1 \le g(x) \le 3n+2\}$ and $f_n = 0$ on $\{x \in X: g(x) \le 3n$ or $g(x) \ge 3(n+1)\}$. If f is defined by $f = \sum_{n=1}^{\infty} f_n$, it is easily verified that $f \in C$. We then have for each $n \in N$ that

$$\phi(f) \ge \sum_{k=1}^{n} \phi(f_k) \ge \sum_{k=1}^{n} \int_{W_k} f_k \, dm \ge \sum_{k=1}^{n} (\alpha_{3k+1} - \alpha_{3k+2})^{-1} m(W_k)$$

$$\ge \sum_{k=1}^{n} (\alpha_{3k+1} - \alpha_{3k+2})^{-1} m(Z_{3k+1} - Z_{3k+2})$$

$$\ge \sum_{k=1}^{n} (\alpha_{3k+1} - \alpha_{3k+2})^{-1} (\alpha_{3k+1} - \alpha_{3k+2}) \ge n.$$

Hence it follows that $\phi(f) = \infty$. This is a contradiction, and the proof is complete.

Definition 4.2. An element $m \in M$ is said to have *compact support* in νX (the realcompactification of X—see [4, p. 116]) if there is a compact set G in νX such that $|m|(X - Z) = 0$ for every zero set Z in νX such that $G \subset Z$. Such a compact set will be called a *compact support* for m in νX. Let M_c denote the subspace of M consisting of all those elements in M which have compact support in νX.

We can now prove the following analogue of Theorem 1.2.

Theorem 4.3. *The order dual of C is isomorphic as a Riesz space to M_c. The isomorphism may be realized by the map $\phi \leftrightarrow m$ where $\phi(f) = \int_X f \, dm$ (where the integral is a Lebesgue integral).*

Proof. Let ϕ be in the order dual of C. Let $\phi = \phi^+ - \phi^-$ be the Jordan decomposition of ϕ. Then by Proposition 4.1, there are $0 \le m_1$, $m_2 \in M_\sigma$ such that $\phi^+(f) = \int_X f \, dm$, and $\phi^-(f) = \int_X f \, dm_2$ for all $f \in C$. By considering the restriction of ϕ to C^b and applying Theorem 1.2, it follows that the map $\phi \to m$ is a Riesz space isomorphism of the order dual of C onto a subspace M^* of M_σ. In order to see that $M_c \subset M^*$, fix $0 \le m \in M_c$ and fix $0 \le f \in C$. If G is a compact support for m in νX, let $Z \equiv \{X \in \nu X: |\overline{f}(x)| \le \|\overline{f}\|_G\}$ where $\overline{f}$ is the continuous extension of f to νX. Since $m(X - Z) = 0$, it follows that $\int_X f \, dm = \int_{X \cap Z} f \, dm \le \|\overline{f}\|_G m(X)$. Thus $\int_X f \, dm < \infty$ for all $0 \le f \in C$. It is now immediate that $M_c \subset M^*$. All that remains to show is that $M_c = M^*$.

Hence let $m \in M^*$ and assume without loss of generality that $0 \le m$. Let

$\phi(f) = \int_X f \, dm$ for all $f \in C$. For each $f \in C$, let $\bar{f}$ denote the unique continuous extension of f to νX. Then the functional $\bar{\phi}$ defined by $\bar{\phi}(\bar{f}) = \phi(f)$ for all $f \in C$ is a nonnegative linear functional on $C(\nu X)$. Since νX is realcompact, it follows by Proposition 4.1 that the Baire measure representing $\bar{\phi}$ has compact support in νX. Let G be the compact set in νX which supports this measure. We will show that G is a compact support for m. Indeed, let Z be any zero set in νX with $G \subset Z$, and define $U = X - Z$. If W is any zero set in X such that $W \subset U$, then $W \cap (X \cap Z) = \emptyset$. Hence there is $g \in C^b$ such that $0 \leq \bar{g} \leq 1$, $\bar{g} = 1$ on W and $\bar{g} = 0$ on Z [4, p. 86]. We then have that $m(W) \leq \int_X g \, dm = \phi(g) = \bar{\phi}(\bar{g}) = 0$. Since $m(W) = 0$ for any zero set $W \subset U$, the regularity of m implies that $m(U) = 0$. The proof is complete.

II. TOPOLOGIES ON THE SPACE L

5. Equicontinuous families. As always X is a completely regular Hausdorff space. A set $B \subset C$ is *equicontinuous* (sometimes called *locally equicontinuous*) if for every positive number ϵ and for every $x \in X$, there is a neighborhood U of x such that $|f(x) - f(y)| \leq \epsilon$ for all $y \in U$ and all $f \in B$. The set B is *pointwise bounded* if $\sup\{|f(x)|: f \in B\}$ is finite for each $x \in X$. The set B is *uniformly bounded* if $\sup\{\|f\|_X : f \in B\}$ is finite. We will let $\mathcal{E}$ denote the family of all pointwise bounded and equicontinuous subsets of C. Let $\mathcal{E}^b$ denote the family of all the sets in $\mathcal{E}$ which are also uniformly bounded. If $B \in \mathcal{E}$, then define $d_B(x, y) = \sup\{|f(x) - f(y)|: f \in B\}$. It is not difficult to verify that the function d_B so defined is a continuous pseudometric on X. It is called the *pseudometric associated with B*. Let p_B be the Minkowski function for B° (where B° denotes the polar of B relative to the pair (L, C)). It is not difficult to verify that $p_B(\xi) = \sup\{|\langle \xi, f \rangle|: f \in B\}$ for every $\xi \in L$. It is clear from this that $d_B(x, y) = p_B(x - y)$ for all $x, y \in X$. The following is a consequence of this observation.

Proposition 5.1. *Let $B \in \mathcal{E}$. Then $B^{\circ\circ} \in \mathcal{E}$, where $B^{\circ\circ}$ is the bipolar of B for the pair (L, C).*

Proof. It is clear that $B^{\circ\circ}$ is pointwise bounded. Let d_B be the pseudometric associated with B, and let p_B be the Minkowski functional of B°. Then $f \in B^{\circ\circ}$ if and only if $|\langle \xi, f \rangle| \leq p_B(\xi)$ for all $\xi \in L$. In particular, if $f \in B^{\circ\circ}$ and $x, y \in X$, then $|f(x) - f(y)| = |\langle x - y, f \rangle| \leq p_B(x - y) = d_B(x, y)$. Since d_B is continuous, if $\epsilon > 0$ and $x \in X$ are given, there is an open neighborhood U of x such that $d_B(x, y) \leq \epsilon$ whenever $y \in U$. It thus follows that $|f(x) - f(y)| \leq \epsilon$ for all $y \in U$ and all $f \in B^{\circ\circ}$. Hence $B^{\circ\circ}$ is equicontinuous as was to be shown.

Proposition 5.2. *Let $B \in \mathcal{E}^b$. Then $B^{\circ\circ} \in \mathcal{E}^b$, where $B^{\circ\circ}$ is the bipolar of B for the pair (L, C^b).*

Proof. Since $B^{\circ\circ}$ is a subset of the bipolar of B for the pair (L, C), $B^{\circ\circ} \in \mathfrak{E}$ by Proposition 5.1. Since B is uniformly bounded, it follows that the associated pseudometric d_B is bounded. Let $d_B(x, y) \leq K$ for all $x, y \in X$ and fix a point $x_0 \in X$. Let M be a number such that $|f(x_0)| \leq M$ for all $f \in B$. Then if $x \in X$ and $f \in B$, it follows that $|f(x)| \leq |f(x) - f(x_0)| + |f(x_0)| \leq d_B(x, x_0) + M \leq K + M$. Hence $B^{\circ\circ}$ is uniformly bounded.

If $B \in \mathfrak{E}$ and $A \subset B$ let f^A be the real-valued function defined by $f^A(x) = \sup\{f(x): f \in A\}$; and let f_A be the real-valued function defined by $f_A(x) = \inf\{f(x): f \in A\}$. In particular, if $A = \{f\}$ then $f^A = f_A = f$. Define $B^* = \{f^A: A \subset B\}$ and $B_* = \{f_A: A \subset B\}$. We then have the following.

Proposition 5.3. *Let $B \in \mathfrak{E}$ ($\mathfrak{E}^b$). Then $B^* \in \mathfrak{E}$ ($\mathfrak{E}^b$) and $B_* \in \mathfrak{E}$ ($\mathfrak{E}^b$).*

Proof. We will show that $B^* \in \mathfrak{E}$ ($\mathfrak{E}^b$). It follows from the definition that B^* is pointwise (uniformly) bounded. In order to prove that B^* is equicontinuous, it is clearly sufficient to show that if $f \in B^*$, then $|f(x) - f(y)| \leq d_B(x, y)$ for all $x, y \in X$. First let $f_1, f_2 \in B$, and let $f = \sup(f_1, f_2)$. Then for all $x, y \in X$, it is clear that

$$|f(x) - f(y)| \leq \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\} \leq d_B(x, y).$$

It then follows by induction that if A is any finite subset of B, then $|f^A(x) - f^A(y)| \leq d_B(x, y)$ for all $x, y \in X$.

In general, let $A \subset B$ and $\epsilon > 0$ be arbitrary. Fix $x, y \in X$. By the definition of f^A, there are $f_1, f_2 \in A$ such that $0 \leq f^A(x) - f_1(x) \leq \epsilon/2$ and $0 \leq f^A(y) - f_2(y) \leq \epsilon/2$. Let $f = \sup(f_1, f_2)$. It is then clear that $0 \leq f^A(x) - f(x) \leq \epsilon/2$ and $0 \leq f^A(y) - f(y) \leq \epsilon/2$. Hence we have that

$$|f^A(x) - f^A(y)| \leq |f^A(x) - f(x)| + |f(x) - f(y)| + |f(y) - f^A(y)| \leq \epsilon + d_B(x, y).$$

Since $\epsilon > 0$ was arbitrary, the result follows.

6. The topology e. Since $C = \bigcup \mathfrak{E}$ and since each set in $\mathfrak{E}$ is $\sigma(C, L)$-bounded, it follows [17, p. 255] that the topology on L of uniform convergence on the sets in $\mathfrak{E}$ is a locally convex topology. This topology will be denoted by e. In this section we will consider some fundamental properties of the space (L, e). The results are due to Raikov [23].

Proposition 6.1. *The dual space of (L, e) is C.*

Proof. Since e is finer than $\sigma(L, C)$, it is clear that C is contained in the dual of (L, e). Now let ϕ be an e-continuous linear functional on L, and let f_0 denote the restriction of ϕ to X. Since ϕ is e-continuous, it is uniformly bounded on B° for some $B \in \mathfrak{E}$. Let M be a bound. For $x \in X$ let U be a

neighborhood of x such that, for all $y \in U$ and all $f \in B$, $|f(x) - f(y)| \leq \epsilon M^{-1}$. Then for all $y \in U$, $M\epsilon^{-1}(x - y) \in B^\circ$ so that $|f_0(x) - f_0(y)| = |\phi(x - y)| \leq M(\epsilon M^{-1})$ $= \epsilon$. Hence f_0 is continuous. Since $\langle \xi, f_0 \rangle = \phi(\xi)$ for all $\xi \in L$, the proof is complete.

The following proposition is easily verified.

Proposition 6.2. *The restriction of e to X is the original topology of X.*

Theorem 6.3. *Let E be a locally convex space and let $\psi: X \to E$ be continuous. Then there is a unique continuous linear transformation T from (L, e) into E such that $T = \psi$ on X.*

Proof. Let T be the unique linear extension of ψ to L, and let V be an arbitrary convex, balanced, closed neighborhood of zero in E. Let V° be the polar of V in the dual of E. Define $B = \{f \in R^X: \exists \phi \in V^\circ, f = \phi \circ \psi\}$. It is clear that $B \subset C(X)$. Also B is pointwise bounded. Indeed, fix $x \in X$. Then for some $n \in N$, $n^{-1}\psi(x) \in V$ so that $|f(x)| = |\phi \circ \psi(x)| \leq n$ for all $f \in B$. Furthermore, B is equicontinuous. Indeed, fix $x \in X$ and $\epsilon > 0$. Then $U = \{\psi^{-1}[\psi(x) + \epsilon V]\} \cap X$ is a neighborhood of x. Also, if $y \in U$ and $f \in B$, then there is $\phi \in V^\circ$ such that $f = \phi \circ \psi$; and hence $|f(x) - f(y)| = |\phi(\psi(x) - \psi(y))| \leq \epsilon$.

It now follows that $T[B^\circ] \subset V^{\circ\circ}$. Indeed, fix $\xi \in B^\circ$, and let $\phi \in V^\circ$ be arbitrary. Then $f = \phi \circ \psi \in B$ so that $|\langle \xi, f \rangle| \leq 1$. Hence

$$
\begin{aligned}
|\phi(T(\xi))| &= \left| \phi\left(\sum\{\xi(x)\psi(x): x \in X\}\right) \right| \\
&= \left| \sum\{\xi(x)\phi \circ \psi(x): x \in X\} \right| = |\langle \xi, f \rangle| \leq 1.
\end{aligned}
$$

Since $V = V^{\circ\circ}$ by the bipolar theorem, it follows that $T[B^\circ] \subset V$, and the proof is complete.

Corollary 6.4. *The topology e is the finest locally convex topology on L whose restriction to X is the original topology of X.*

7. The topology e^b. Since $C^b = \bigcup \mathcal{E}^b$ and since every set in $\mathcal{E}^b$ is $\sigma(C^b, L)$-bounded, the topology of uniform convergence on the sets in $\mathcal{E}^b$ is a locally convex topology on L. This topology will be denoted by e^b. In this section, we will establish properties of e^b analogous to those of e above.

Proposition 7.1. *The dual of (L, e^b) is C^b.*

Proof. Since e^b is finer than $\sigma(L, C^b)$, it is clear that C^b is a subset of the dual space. Now let ϕ be an e^b-continuous linear functional on L. Then ϕ is e-continuous so that by Proposition 6.1, there is an $f \in C$ such that

$\phi(\xi) = \langle \xi, f \rangle$ for all $\xi \in L$. It must be shown that f is uniformly bounded. Let $B \in \mathfrak{E}^b$ be such that ϕ is bounded on B°. Thus there is a number M such that $|\langle \xi, f \rangle| \le M$ for all $\xi \in B^\circ$. Let K be a uniform bound for all of the functions in B. Then $K^{-1}x \in B^\circ$ for all $x \in X$ so that $|f(x)| \le K \cdot M$ for all $x \in X$. The proof is complete.

Proposition 7.2. *The restriction of e^b to X is the original topology of X.*

Theorem 7.3. *Let E be a locally convex space and let $\psi : X \to E$ be continuous. If $\psi[X]$ is bounded, then there is a unique continuous linear transformation from (L, e^b) into E such that $T = \psi$ on X.*

Proof. Let T be the unique linear extension of ψ to L. Let V be a closed, balanced and convex neighborhood of zero in E, and let V° be the polar of V in the dual of E. Now define $B = \{f \in R^X : \exists \phi \in V^\circ, f = \phi \circ \psi\}$. Since $\psi[X]$ is bounded in E, it follows that $B \subset C^b$. Just as in Proposition 6.3, it follows that $B \in \mathfrak{E}$. Now take a positive number K such that $\psi[X] \subset M \cdot V$. (This is possible since $\psi[X]$ is bounded.) This means that $|f(x)| = |\phi(\psi(x))| \le M$ for all $f \in B$. Hence $B \in \mathfrak{E}^b$. Finally, as in the proof of Theorem 6.3, $T[B^\circ] \subset V^{\circ\circ} = V$.

Corollary 7.4. *The topology e^b is the finest locally convex topology on L whose restriction to X is the original topology of X and for which X is a bounded set.*

Proof. In view of Theorem 7.3, it is sufficient to show that X is bounded in (L, e^b). But this is immediate since the restriction to X of every continuous linear functional on (L, e) is bounded by Proposition 7.1.

In the remainder of this section, we will discuss the topology on L of uniform convergence on the norm compact subsets if C^b. This topology will be denoted by s. The basic properties of the s topology are consequences of the following proposition.[2]

Proposition 7.5. *The space (L, s) is a dense subspace of $(L(\beta X), e^b(\beta X))$.*

Proof. If $B \subset C^b$, let $\overline{B} = \{\overline{f} : f \in B\}$ where f is the continuous extension of f to βX. By the Ascoli-Arzelá theorem, B is norm compact if and only if $\overline{B} \in e^b(\beta X)$. Hence s is the restriction of $e^b(\beta X)$ to L. From Proposition 7.1, it is immediate that L is dense in $(L(\beta X), e^b(\beta X))$.

The following is an immediate consequence of Propositions 7.1 and 7.5.

---

[2] The author would like to thank the referee for formulating Proposition 7.5 and for indicating the simplifications it affords to studying the s topology (in particular in the proofs of Theorems 7.8, 8.14 and 8.16.)

Proposition 7.6. *The dual space of* (L, s) *is* C^b.

Since X is dense in βX, the following is immediate from Propositions 7.2 and 7.5.

Proposition 7.7. *The restriction of s to X is the original topology of X.*

Theorem 7.8. *Let E be a locally convex space and let $\psi\colon X \to E$ be continuous. If $\psi[X]$ is totally-bounded, then there is a unique continuous linear transformation T from (L, s) into E such that $T = \psi$ on X.*

Proof. Let $\hat{E}$ denote the completion of E. Then $\psi[X]$ is relatively compact in $\hat{E}$. Let $\psi\colon \beta X \to \hat{E}$ be the continuous extension of ψ to βX. By Theorem 7.3 there is a continuous linear transformation $\hat{T}$ from $(L(\beta X), e^b(\beta X))$ into $\hat{E}$ such that $\hat{T} = \hat{\psi}$ on βX. Let T denote the restriction of $\hat{T}$ to L. By Proposition 7.5, T is a continuous linear transformation of (L, s) into E. The uniqueness of T is immediate.

Corollary 7.9. *The topology s is the finest locally convex topology on L for which X is totally bounded.*

Proof. Since X is relatively compact in $(L(\beta X), e^b(\beta X))$, it follows from Proposition 7.5 that X is totally bounded in (L, s). The result is now immediate from Theorem 7.8.

III. COMPLETIONS OF L

In this part of the paper, we will investigate the structure of the completion of the space L under several different topologies. Each of the main completion results (Theorems 8.7, 8.15 and 9.5) in §§8 and 9 are generalizations of the Katětov-Pták theorem. A crucial tool in these investigations is the following theorem of A. Grothendieck which we will refer to as the *completion theorem.* (See [17, p. 270].)

Theorem. *Let E be a locally convex Hausdorff space, let E' denote the dual space of E and let $\mathfrak{U}$ denote a basis for the neighborhood system at zero in E. The completion of E is the space $\hat{E}$ of all linear functionals on E' which are $\sigma(E', E)$-continuous on U° for every $U \in \mathfrak{U}$. The topology of $\hat{E}$ is that of uniform convergence on the sets in the family $\{U^\circ : U \in \mathfrak{U}\}$.*

8. The completion of (L, e^b). Denote the completion of (L, e^b) by $\hat{L}^b$. We then have the following.

Proposition 8.1. $\hat{L}^b$ *is a subspace of the Banach dual of* C^b.

Proof. If $\phi \in \hat{L}^b$, it then follows by Proposition 7.1 and the completion

theorem that ϕ is a linear functional on C^b. Let $\{f_n\}$ be a sequence in C^b such that $f_n \to 0$ uniformly. Then $\{f_n\} \in \mathcal{E}^b$. Since the family $\mathcal{U} = \{B^\circ: B \in \mathcal{E}^b\}$ is a local basis at zero for e^b and since $\{f_n\}$ is a sequence in $\{f_n\}^{\circ\circ}$ with $f_n \to 0$ in the $\sigma(C^b, L)$-sense (i.e. pointwise), it follows from the completion theorem that $\phi(f_n) \to 0$. Hence ϕ is in the Banach dual of C^b. The proof is complete.

As a consequence of Proposition 8.1 and Theorem 1.2, we may identify $\hat{L}^b$ with a subspace of M. The set functions in this subspace have some rather interesting continuity properties (which are, in a sense, generalized dominated convergence and monotone convergence results) as the following proposition shows.

Proposition 8.2. *Let* m *belong to* M. *Then the following are equivalent*:
1. $m \in \hat{L}^b$.
2. *Let* $\{f_i\} \in \mathcal{E}^b$ *be a net such that* $f_i \to f$ *pointwise on* X. *Then* $\int_X f_i \, dm \to \int_X f \, dm$.
3. *Let* $\{f_i\} \in \mathcal{E}^b$ *be a downward directed net with* $f_i \downarrow f$ *pointwise on* X. *Then* $\int_X f_i \, dm \to \int_X f \, dm$.
4. *Let* $\{f_i\} \in \mathcal{E}^b$ *be a downward directed net with* $f_i \downarrow 0$ *pointwise on* X. *Then* $\int_X f_i \, dm \to 0$.
5. *Let* $\{f_i\} \in \mathcal{E}^b$ *be a net such that* $f_i \to f$ *pointwise on* X. *Then* $\int_X f_i \, d|m| \to \int_X f \, d|m|$.

Proof. ($1 \Rightarrow 2$) This is immediate from the completion theorem since $\{f_i\}^\circ$ is an e^b-neighborhood of zero and since $\{f_i\}^{\circ\circ} \in \mathcal{E}^b$ by Proposition 5.2.

($2 \Rightarrow 1$) Since $\{B^\circ: B \in \mathcal{E}^b\}$ is a local base at zero for e^b and since $B^{\circ\circ} \in \mathcal{E}^b$ whenever $B \in \mathcal{E}^b$ by Proposition 5.2, the result is immediate from the completion theorem.

($2 \Rightarrow 3$) This is obvious.

($3 \Rightarrow 4$) This is obvious.

($4 \Rightarrow 5$) Let $\{f_i\} \in \mathcal{E}^b$ be a net such that $f_i \to f$ pointwise. Without loss of generality, assume that $f = 0$. For each i, define $g_i = \sup\{f_j: i \le j\}$. Then $\{g_i\} \in \mathcal{E}^b$ by Proposition 5.3. Also $\{g_i\}$ is a downward directed net with $g_i \downarrow 0$ pointwise on X.

We will now show that $\int_X g_i \, d|m| \to 0$, or, equivalently, that $\int_X g_i \, dm^+ \to 0$ and $\int_X g_i \, dm^- \to 0$. Since the argument is the same for both, we will only show $\int_X g_i \, dm^+ \to 0$. If this is not so, by passing to a subnet if necessary, we may assume that there is a positive number α such that $0 < 3\alpha \le \int_X g_i \, dm^+$ for all i.

Since $\{g_i\} \in \mathcal{E}^b$ there is a uniform bound K for the family $\{g_i\}$. By Proposition 1.3, there are disjoint zero sets Z_1 and Z_2 in X with $m^+(X - Z_1) \le \alpha K^{-1}$ and $m^-(X - Z_2) \le \alpha K^{-1}$. Let $g \in C^b$ be such that $0 \le g \le 1$, $Z_1 = \{x \in X: g(x) = 1\}$ and $Z_2 = \{x \in X: g(x) = 0\}$. Then the set of functions $\{gg_i\}$ belongs to $\mathcal{E}^b$

as can easily be verified. Since $\{gg_i\}$ is a downward directed net with $gg_i \downarrow 0$ pointwise on X, it follows by assumption that $\int_X gg_i\,dm \rightarrow 0$. On the other hand, we have the following:

$$\int_X gg_i\,dm = \int_X gg_i\,dm^+ - \int_X gg_i\,dm^-$$

$$= -\int_{X-Z_1}(1-g)g_i\,dm^+ + \int_X g_i\,dm^+ - \int_{X-Z_2}gg_i\,dm^-$$

$$\geq -Km^+(X-Z_1) + 3\alpha - Km^-(X-Z_2)$$

$$\geq -K(\alpha K^{-1}) + 3\alpha - K(\alpha K^{-1}) \geq \alpha > 0.$$

Since the above inequality holds for all i, it contradicts the fact that $\int_X gg_i\,dm \rightarrow 0$. Hence $\int_X g_i\,d|m| \rightarrow 0$ as asserted.

In order to complete the proof consider the nets $\{f_i^+\}$ and $\{f_i^-\}$. Each of these belongs to $\mathcal{E}^b$ by Proposition 5.3, and each converges pointwise to zero on X since $f_i \rightarrow 0$ pointwise. If for each i, $g_i^+ = \sup\{f_j^+ : i \leq j\}$ and $g_i^- = \sup\{f_j^- : i \leq j\}$, then we have shown above that $\int_X g_i^+\,d|m| \rightarrow 0$ and $\int_X g_i^-\,d|m| \rightarrow 0$. Since $0 \leq \int_X f_i^+\,d|m| \leq \int_X g_i^+\,d|m|$ and $0 \leq \int_X f_i^-\,d|m| \leq \int_X g_i^-\,d|m|$, it follows that $\int_X f_i\,d|m| = \int_X f_i^+\,d|m| - \int_X f_i^-\,d|m| \rightarrow 0$. This is what was to be shown.

$(5 \Rightarrow 2)$ Let $\{f_i\} \in \mathcal{E}^b$ be a net such that $f_i \rightarrow f$ pointwise on X. Assume without loss of generality that $f = 0$. Then by Proposition 5.2, $\{|f_i|\} \in \mathcal{E}^b$ is a net such that $|f_i| \rightarrow 0$ pointwise on X. Hence, applying the assumption,

$$\left|\int_X f_i\,dm\right| \leq \int_X |f_i|\,d|m| \rightarrow 0.$$

The proof of the proposition is complete

Proposition 8.3. *The space $\hat{L}^b$ is an ideal in the Riesz space M.*

Proof. We first show that $\hat{L}^b$ is a sublattice. To do this it is sufficient to prove that if $m \in \hat{L}^b$, then $|m| \in \hat{L}^b$. But if $m \in \hat{L}^b$, then $|m|$ satisfies condition 5 of Proposition 8.2. But then the implication $(2 \Rightarrow 1)$ in Proposition 8.2 shows that $|m| \in \hat{L}^b$. In order to see that $\hat{L}^b$ is an ideal, let m_1, $m_2 \in M$ be such that $|m_1| \leq |m_2|$ and $m_2 \in \hat{L}^b$. Let $\{f_i\} \in \mathcal{E}^b$ be a downward directed net with $f_i \downarrow 0$ pointwise on X. Then $\int_X f_i\,d|m_2| \rightarrow 0$ by condition 5 of Proposition 8.2. Since $|\int_X f_i\,dm_1| \leq \int_X f_i\,d|m_1| \leq \int_X f_i\,d|m_2|$, it follows that $\int_X f_i\,dm_1 \rightarrow 0$. Hence $m_1 \in \hat{L}^b$ by the implication $(4 \Rightarrow 1)$ of Proposition 8.2. The proof is complete.

Proposition 8.4. *The space $\hat{L}^b$ is a subspace of M_σ.*

Proof. Let $m \in \hat{L}^b$. By Proposition 1.6, it is sufficient to prove that if

$\{f_n\} \subset C^b$ is a decreasing sequence with $f_n \downarrow 0$ pointwise on X, then $\int_X f_n \, dm$ $\to 0$. But it is an easy exercise to verify that if $f_n \downarrow 0$ pointwise, then $\{f_n\} \in \mathfrak{G}^b$. The result is now immediate from Proposition 8.2.

The following result gives a necessary and sufficient condition for a non-negative element of M_σ to belong to M_s. Other equivalent conditions are given by Sentilles and Wheeler in [24]. The proof given below is similar to that outlined in their work. (The author would like to take this opportunity to thank Professor Sentilles for sending him a copy of this work while it was still in progress.)

Proposition 8.5. *Let* $0 \le m \in M_\sigma$. *Then* $m \in M_s$ *if and only if for every net* $\{f_i\} \in \mathfrak{G}^b$ *with* $f_i \to f$ *pointwise on* X, *it follows that* $\int_X f_i \, dm \to \int_X f \, dm$.

Proof. ($\Rightarrow$) Let $\{f_i\} \in \mathfrak{G}^b$ be such that $f_i \to f$ pointwise on X. Without loss of generality, assume that $f = 0$. If it is not true that $\int_X f_i \, dm \to 0$, then, by passing to a subnet if necessary, there is a positive number α such that $\int_X f_i \, dm \ge \alpha > 0$ for all i. Define $d(x, y) = \sup_i |f_i(x) - f_i(y)|$ for all $x, y \in X$. Then d is a continuous pseudometric on X. Since $m \in M_s$, there is a zero set Z_d in X such that $m(X - Z_d) = 0$, Z_d is d-closed and Z_d contains a countable d-dense set Y. Since Y is countable, there is a sequence $\{g_n\}$ in $\{f_i\}$ such that $g_n(y) \to 0$ as $n \to \infty$ for all $y \in Y$.

It now follows that $g_n(x) \to 0$ as $n \to \infty$ for all $x \in Z_d$. Indeed, fix $x \in Z_d$ and $\epsilon > 0$. Take $y \in Y$ such that $d(x, y) \le \epsilon/2$. Choose a natural number n_0 such that $|g_n(y)| \le \epsilon/2$ whenever $n \ge n_0$. Then for $n \ge n_0$ it follows that $|g_n(x)| \le |g_n(y)| + |g_n(y) - g_n(x)| \le \epsilon/2 + d(x, y) \le \epsilon$. Since $x \in X$ and $\epsilon > 0$ were arbitrary, $g_n \to 0$ pointwise on Z_d as claimed. But $m(X - Z_d) = 0$ so that $g_n \to 0$ pointwise almost everywhere on X for the measure m. By the dominated convergence theorem it follows that $\int_X g_n \, dm \to 0$ as $n \to \infty$. But this contradicts the fact that $\int_X f_i \, dm \ge \alpha > 0$ for all i.

($\Leftarrow$) Let d be an arbitrary continuous pseudometric on X. Let m_d be the set function induced by m on the associated metric space (X^*, d^*) according to Proposition 3.5. Since $m \in M_\sigma$ it follows by Proposition 3.5 that m_d is a Baire measure. Since a metric space is in particular paracompact, if m_d is net-additive, it then follows by Proposition 2.3 that m_d is supported on a d-closed, Lindelöf subspace Z^* of X^*. Also Z^* is d-separable since any Lindelöf subspace of a metric space is separable. Then $Z = \Theta^{-1}[Z^*]$ is a zero set in X which is d-closed and d-separable (where Θ is the map of Lemma 3.5) and which is such that $m(X - Z) = 0$. Thus all that need be shown is that m_d is net-additive on X^* for every continuous pseudometric on X.

Assume that m_d is not net-additive for some continuous pseudometric on X. We will show that we may assume without loss of generality that m_d has an empty

support. Since m_d is not net-additive (as is shown in Pym [21]), there is a non-zero Baire measure μ^* on X^* with $0 < \mu^* \leq m_d$ which has empty support. Let E be the subspace of $C^b(X)$ consisting of those functions f such that $f = f^* \circ \Theta$ for some $f^* \in C^b(X^*)$. Define a functional ϕ^* on E by $\phi^*(f) = \int_{X^*} f^* d\mu^*$ for all $f \in E$. Define a seminorm p on $C^b(X)$ by $p(f) = \int_X f^+ dm$ for all $f \in C^b$. It is clear that $\phi^*(f) \leq p(f)$ for all $f \in E$. By the Hahn-Banach theorem, there is an extension ϕ of ϕ^* such that $\phi(f) \leq p(f)$ for all $f \in C^b$. It is simple to verify that $0 \leq \phi$. Let $0 \leq \mu \in M$ represent ϕ according to 1.2. Then for $0 \leq f \in C^b$, $\int_X f d\mu \leq p(f) = \int_X f dm$. That is $0 \leq \mu \leq m$. Since $m \in M_\sigma$, it follows that $\mu \in M_\sigma$. Furthermore, if $\{f_i\} \in \mathfrak{S}^b$ is a net such that $f_i \to f$ pointwise on X, then $|\int_X (f - f_i) d\mu| \leq \int_X |f - f_i| d\mu \leq \int_X |f - f_i| dm \to 0$. Finally, since ϕ extends ϕ^*, $\mu_d = \mu^*$. Hence μ satisfies the same conditions (relative to the proof) as m, and in addition has the property that μ_d has empty support.

As shown above we may assume that m_d is entirely without support. This means that for each point $\xi \in X^*$, there is an open set U_ξ containing ξ such that $m_d(U_\xi) = 0$. Since $\{U_\xi : \xi \in X^*\}$ is an open cover of X^*, there is a partition of unity $\{f_i^* : i \in I\}$ subordinate to this cover. (Hence, in particular, $\int_{X^*} f_i^* dm_d = 0$ for all $i \in I$.) Define $f_i = f_i^* \circ \Theta$, and let $\mathfrak{S}$ denote the family of all finite subsets of I. For $\sigma \in \mathfrak{S}$ define $g_\sigma = \Sigma\{f_i : i \in \sigma\}$. Then $\{g_\sigma\} \in \mathfrak{S}^b$. (This is true since $\{f_i^*\}$ being a partition of unity implies that there is an open set about each point in X in which all but a finite number of the functions f_i are zero.) Furthermore $g_\sigma \to 1$ pointwise on X. Hence by assumption $\int_X g_\sigma dm \to m(X) > 0$. However $\int_X g_\sigma dm = \Sigma\{\int_{X^*} f_i^* dm_d : i \in \sigma\} = 0$ for all $\sigma \in \mathfrak{S}$. This is a contradiction, and the proof is complete.

Theorem 8.6. *The completion of* (L, e^b) *is the space* M_s *equipped with the topology of uniform convergence on the sets in* $\mathfrak{S}^b$.

Proof. We will begin by showing that $\hat{L}^b$ is contained in M_s. By Proposition 8.3 it is sufficient to prove that if $0 \leq m \in \hat{L}^b$, then $m \in M_s$. However, this is immediate from the implication $(1 \Rightarrow 2)$ of Proposition 8.2 and Proposition 8.5. Again, to show that M_s is contained in $\hat{L}^b$ (since M_s is a lattice), it is enough to show that if $0 \leq m \in M_s$, then $m \in \hat{L}^b$. But this follows from Proposition 8.5 and the implication $(2 \Rightarrow 1)$ of Proposition 8.2. Hence $\hat{L}^b = M_s$. It follows from the completion theorem and Proposition 5.2 that the topology on M_s as the completion of (L, e^b) is the topology of uniform convergence on the sets in $\mathfrak{S}^b$. The proof is complete.

If X is a D-space (Definition 3.3), then $\hat{L}^b = M_s - M_\sigma$. As noted above, it is consistent with the axioms of set theory to assume that every space is a D-space. Hence it is consistent to assume that $\hat{L}^b = M_\sigma$ for every completely

regular Hausdorff space X. The main facts are summarized in the following theorem.

Theorem 8.7. *Let X be a completely regular Hausdorff space (a D-space) and let $M_s(M_\sigma)$ be endowed with the topology of uniform convergence on the uniformly bounded and equicontinuous subsets of C^b. Then the following hold:*

1. *$M_s(M_\sigma)$ is complete.*
2. *L is dense in $M_s(M_\sigma)$.*
3. *The dual space for $M_s(M_\sigma)$ is C^b.*
4. *X is a bounded subset of $M_s(M_\sigma)$.*
5. *The restriction of the topology to X is the original topology of X.*

Proof. Both 1 and 2 are immediate from Theorem 8.6. That 3 holds follows from Proposition 7.1 and the fact that the completion of a locally convex space has the same dual as the original space. Numbers 4 and 5 are just restatements of Proposition 7.2 and Corollary 7.4.

The following result was proved by Granirer in [6].

Corollary 8.8. *Let X be a completely regular Hausdorff space (a D-space).*

1. *If B is a uniformly bounded and equicontinuous subset of C^b, then B is relatively $\sigma(C^b, M_s)$-compact ($\sigma(C^b, M_\sigma)$-compact).*

2. *If X is not a D-space, then there is a uniformly bounded and equicontinuous family in C^b which is not relatively $\sigma(C^b, M_\sigma)$-compact.*

Proof. (1) This is immediate from statement 3 of Theorem 8.7 and the Banach-Alaoglu theorem.

(2) If X is not a D-space, $M_s \neq M_\sigma$. Let M_σ have the topology of uniform convergence on the sets in $\mathcal{E}^b$. By Theorem 8.7 M_s is a closed subspace of M_σ. Hence by the Hahn-Banach theorem, there is a nonzero continuous linear functional on M_σ which vanishes on M_s. This means that the dual space of M_σ is strictly larger than C^b. The result is now immediate from the Mackey-Arens theorem.

Remarks. 1. If X is compact (or pseudocompact), then Theorem 8.6 reduces to the Katětov-Pták theorem.

2. If X is discrete, then $\mathcal{E}^b$ is coextensive with the family of uniformly bounded sets of bounded, real-valued functions on X. In the case, $M_s = l^1(X)$ with its norm topology.

In view of Remark 2, it is natural to ask if there are any other cases in which M_s is normable. The following provides a negative answer to the question.

Theorem 8.9. *Let X be a completely regular Hausdorff space, and let M_s have the topology of uniform convergence on the uniformly bounded, equicontinuous subsets of C^b. Then the following are equivalent:*

1. X is discrete.
2. M_s is normable.
3. M_s is metrizable.
4. (L, e^b) is a Mackey space. (That is, the topology e^b on L is the Mackey topology for the pair (L, C^b).)

Proof. ($1 \Rightarrow 2$) This is the content of Remark 2 above.

($2 \Rightarrow 3$) This is obvious.

($3 \Rightarrow 4$) This is a consequence of the fact that any metrizable locally convex space is a Mackey space.

($4 \Rightarrow 1$) Let $m(L, C^b)$ and $m(M, C^b)$ denote the Mackey topologies for the pairs (L, C^b) and (M, C^b) respectively. Let τ and σ denote the restrictions to L of the topologies $m(M, C^b)$ and $\sigma(M, C^b)$ respectively. Since $L \subset M$, it is not difficult to verify that (*) $m(L, C^b) \supset \tau \supset \sigma$. By assumption $m(L, C^b)$ is the topology of uniform convergence on the sets in $\mathcal{E}^b$ so that by Theorem 8.7(5), the restriction of $m(L, C^b)$ to X is the original topology of X. Since it is well known that the restriction to X of $\sigma(M, C^b)$ is the original topology of X, it follows that the restriction of $m(M, C^b)$ to X is the original topology of X. However, in [15] it is shown that the restriction of $m(M, C^b)$ to X is the original topology of X if and only if X is discrete. The proof is complete.

We will now discuss the question of when $M_\sigma^+ = \{m \in M_\sigma : 0 \leq m\}$ is metrizable for the topology of uniform convergence on the sets in $\mathcal{E}^b$. For this purpose we recall two important theorems from the literature. The first is due to Granirer in [6].

Theorem 8.10. Let X be a D-space. If $\{m_i\} \subset M_\sigma^+$ is a net and if $\{m_i\}$ converges to $m \in M_\sigma^+$ in the $\sigma(M_\sigma, C^b)$-sense, then $m_i \to m$ for the topology of uniform convergence on the uniformly bounded and equicontinuous subsets of C^b.

Note that Theorem 8.10 implies that if X is a D-space, then on M_σ^+ the topology of uniform convergence on the sets $\mathcal{E}^b$ is identical with $\sigma(M_\sigma, C^b)$. The next theorem is due to Varadarajan in [25].

Theorem 8.11. Let (X, d) be a metric space. Then $M_\tau^+ = \{m \in M_\tau : 0 \leq m\}$ is metrizable for the weak topology $\sigma(M_\tau, C^b)$.

We can now prove the following.

Theorem 8.12. Let X be a D-space. Then X is metrizable if and only if M_σ^+ is metrizable for the topology of uniform convergence on the uniformly bounded and equicontinuous sets in C^b.

Proof. Since $X \subset M_\sigma^+$, it follows from condition (5) of Theorem 8.7 that if

M_σ^+ is metrizable, then X is also metrizable. Now assume that X is metrizable. Then by Proposition 2.2, $M_\tau = M_\sigma$. By Varadarajan's result (Theorem 9.11), the topology $\sigma(M_\sigma, C^b)$ on M_σ^+ is metrizable. By Granirer's result (Theorem 8.10), the topology of uniform convergence on the sets in $\mathcal{E}^b$ is identical with $\sigma(M_\sigma, C^b)$ on M_σ^+. Hence the result follows.

Finally we include the following result due to Dudley [3].

Theorem 8.13. *Let X be completely regular Hausdorff, and let M_s be endowed with the topology of uniform convergence on the pointwise bounded and equicontinuous sets in C^b. A set $A \subset M_s$ is relatively sequentially compact if and only if it is weakly relatively sequentially compact.*

Proof. In [3], Dudley proves that a sequence $\{m_n\} \subset M_s$ converges weakly to $m \in M_s$ if and only if it converges uniformly on the sets in $\mathcal{E}^b$. The result is the immediate.

The remainder of this section will be devoted to a discussion of the space (L, s).

Theorem 8.14. *The completion of (L, s) is the space M equipped with the topology of uniform convergence on the norm compact subsets of C^b.*

Proof. By Theorem 8.6 the completion of $(L(\beta X), e^b(\beta X))$ is $M_s(\beta X) = M(X)$. But (L, s) is dense in $(L(\beta X), e^b(\beta X))$ so that the result follows immediately.

The basic results about the s topology are summarized in the following.

Theorem 8.15. *Let X be a completely regular Hausdorff space and let M be equipped with the topology of uniform convergence on the norm compact subsets of C^b. Then the following hold:*
1. *M is complete.*
2. *L is dense in M.*
3. *The dual space of M is C^b.*
4. *X is a totally bounded subset of M.*
5. *The restriction of the topology on M to X is the original topology of X.*

Remark. If X is compact, then Theorem 8.15 reduces to the Katětov-Pták theorem via the Ascoli-Arzelá theorem.

Theorem 8.16. *Let X be a completely regular Hausdorff space and let M be endowed with the topology of uniform convergence on the norm compact subsets of C^b. Then the following are equivalent:*
1. *X is finite.*
2. *M is normable.*
3. *M is metrizable.*

Proof. We only need to show that 3 implies 1. But from Proposition 7.5, if M is metrizable, then $M_s(\beta X)$ is metrizable (for the topology of uniform convergence on the uniformly bounded, equicontinuous subsets of $C(\beta X)$): Hence by Theorem 8.9, βX is discrete. But this implies that X is finite.

9. **The completion of** (L, e). Let $\hat{L}$ denote the completion of (L, e). We then have the following.

Proposition 9.1. $\hat{L}$ *is a subspace of the order dual of* C.

Proof. If $\phi \in \hat{L}$, then it follows from Proposition 6.1 and the completion theorem that ϕ is a linear functional on C. If ϕ does not belong to the order dual of C, then there is a function $f \in C$ with $0 \leq f$ and a sequence $\{g_n\} \subset C$ such that $0 \leq g_n \leq f$ and such that $0 < |\phi(g_n)| = \alpha_n \uparrow +\infty$. Now define $b_n = \alpha_n^{-\frac{1}{2}} g_n$ for all $n \in N$. Then $b_n \to 0$ in the $\sigma(C, L)$-sense (i.e. pointwise), and $\{b_n\}$ is pointwise bounded. Furthermore, $\{b_n\}$ is equicontinuous. Indeed, let $x \in X$ and let ϵ be a positive number. Choose $n_0 \in N$ such that $\alpha_n^{-\frac{1}{2}} \leq \epsilon(\epsilon + f(x))^{-1}$ for all $n \geq n_0$. Let U be a neighborhood of x such that if $y \in U$, then (1) $|f(x) - f(y)| \leq \epsilon$ and (2) $|b_n(x) - b_n(y)| \leq \epsilon$ for $n = 1, 2, \cdots, n_0$. Now observe, since $0 \leq g_n \leq f$ for all $n \in N$, that $|g_n(x) - g_n(y)| \leq \epsilon + f(x)$ for all $y \in U$ by (1). Hence if $n \geq n_0$, we have for all $y \in U$ that $|b_n(x) - b_n(y)| = \alpha_n^{-\frac{1}{2}} |g_n(x) - g_n(y)| \leq \epsilon(\epsilon + f(x))^{-1}(\epsilon + f(x)) = \epsilon$. On the other hand, if $n = 1, \cdots, n_0$, then for all $y \in U$ we have by 2 that $|b_n(x) - b_n(y)| \leq \epsilon$. Thus we have proved that $\{b_n\} \in \mathfrak{E}$. Since $\phi \in \hat{L}$ and since $b_n \to 0$ pointwise, it follows from the completion theorem that $\phi(b_n) \to 0$. However, $|\phi(b_n)| = \alpha_n^{-\frac{1}{2}} |\phi(g_n)| = \alpha_n^{\frac{1}{2}} \nearrow +\infty$. This is a contradiction, and the proof is complete.

As a consequence of Theorem 4.4 and Proposition 9.1 we may identify $\hat{L}$ with a subspace of M_c. The following gives some more information about this subspace.

Proposition 9.2. $\hat{L}$ *is a subspace of* M_s.

Proof. If $\phi \in \hat{L}$, it then follows by Proposition 9.1 that ϕ is a linear functional on C. Since $\{B^\circ : B \in \mathfrak{E}\}$ is a local base at zero for e, it follows by the completion theorem and by Proposition 5.1 that ϕ is $\sigma(C, L)$-continuous on each $B \in \mathfrak{E}$. In particular, the restriction of ϕ to C^b is $\sigma(C^b, L)$-continuous on each $B \in \mathfrak{E}^b$ so that the restriction of ϕ to C^b belongs to the completion of (L, e^b) by the completion theorem. Hence by Theorem 8.6, there is an $m \in M_s$ such that $\phi(f) = \int_X f\, dm$ for all $f \in C^b$. But then by Proposition 9.1 and Theorem 4.4, it is immediate that m is the element in M_c which represents ϕ on C. The proof is complete.

Proposition 9.3. *Let* $0 \leq m \in M$. *Then the following are equivalent:*

1. $m \in \hat{L}$.

2. *If* $\{f_i\} \in \mathfrak{E}$ *is a net with* $f_i \to f$ *pointwise on* X, *then* $\int_X f_i \, dm \to \int_X f \, dm$.

3. *If* $\{f_i\} \in \mathfrak{E}$ *is a downward directed net with* $f_i \downarrow f$ *pointwise on* X, *then* $\int_X f_i \, dm \to \int_X f \, dm$.

4. *If* $\{f_i\} \in \mathfrak{E}$ *is a downward directed net with* $f_i \downarrow 0$, *then* $\int_X f_i \, dm \to 0$.

Proof. The proof of the implications $(1 \leftrightarrow 2 \Rightarrow 3 \Rightarrow 4)$ is just as in Proposition 8.2.

$(4 \Rightarrow 2)$ Let $\{f_i\} \in \mathfrak{E}$ be a net with $f_i \to f$ pointwise on X and assume $f = 0$. By Proposition 5.3 we may assume $0 \leq f_i$ for all i. (Otherwise consider $\{f_i^+\}$ and $\{f_i^-\}$ separately.) For each i define $g_i = \sup\{f_j : i \leq j\}$. Then by Proposition 5.3, $\{g_i\} \in \mathfrak{E}$ is a downward directed net with $g_i \downarrow 0$. Hence by assumption $\int_X g_i \, dm \to 0$. Since $0 \leq \int_X f_i \, dm \leq \int_X g_i \, dm$ for all i, the proof is complete.

We will now represent the completion of the space (L, e). For this purpose, define $M_{sc} = M_c \cap M_s$. Then M_{sc} is an ideal in the Riesz space M.

Theorem 9.4. *The completion of the space* (L, e) *is the space* M_{sc} *equipped with the topology of uniform convergence on the sets in* $\mathfrak{E}$.

Proof. It is a consequence of Proposition 9.1, Proposition 9.2 and Theorem 4.4 that $\hat{L}$ is a subspace of M_{sc}. In order to prove the reverse inclusion, take $m \in M_{sc}$, and assume without loss of generality that $0 \leq m$. Since $\{B^\circ : B \in \mathfrak{E}\}$ is a local base at zero for e, it is sufficient by Proposition 9.3 to prove that if $\{f_i\} \in \mathfrak{E}$ is downward directed net with $f_i \downarrow 0$ pointwise on X, then $\int_X f_i \, dm \to 0$. For each $f \in C$, let $\bar{f}$ denote the unique continuous extension of f to νX. Define a nonnegative functional $\bar{\phi}$ on $C(\nu X)$ by $\bar{\phi}(\bar{f}) = \int_X f \, dm$ for all $f \in C$. (The definition makes sense by Proposition 4.3(b).) Then by Theorem 4.4, there is a regular Baire measure $\bar{m}$ on νX which has a compact set $G \subset \nu X$ as its support and which is such that $\bar{\phi}(\bar{f}) = \int_{\nu X} \bar{f} \, d\bar{m}$ for all $f \in C$. If we can show that $\bar{f}_i \downarrow 0$ pointwise on G, it will follow by Dini's theorem that $\bar{f}_i \downarrow 0$ uniformly on G. Hence $\phi(f_i) = \bar{\phi}(\bar{f}_i) = \int_{\nu X} \bar{f}_i \, d\bar{m} = \int_G \bar{f}_i \, d\bar{m} \to 0$. (Since $\bar{m}$ has compact support, it is net-additive and so a Borel measure by Proposition 1.8. Hence $\int_G \bar{f}_i \, d\bar{m}$ makes sense.)

We will now show that $\bar{f}_i \downarrow 0$ pointwise on G. Define $d(x, y) = \sup\{|f_i(x) - f_i(y)|\}$. Then d is a continuous pseudometric on X. Since $m \in M_s$, there is a d-closed zero set Z_d in X such that $m(X - Z_d) = 0$ and a countable, d-dense subset Y in Z_d. Let $\{g_n\}$ be a decreasing sequence in $\{f_i\}$ with $g_n \downarrow 0$ pointwise on Y. Then $g_n \downarrow 0$ pointwise on Z_d. Indeed, take $x \in Z_d$ and $\epsilon > 0$. Choose $y \in Y$ with $d(x, y) \leq \epsilon/2$. Then take n_0 so large that for

$n \geq n_0$, $g_n(y) \leq \epsilon/2$. Hence for $n \geq n_0$, $|g_n(x)| \leq |g_n(x) - g_n(y)| + |g_n(y)| \leq d(x, y)$ $+ \epsilon/2 \leq \epsilon$. Thus $g_n \downarrow 0$ on Z_d as asserted.

Let Z^* be any zero set in νX with $X \cap Z^* = Z_d$. Then $G \subset Z^*$. Indeed, let W be any zero set in $\nu X - Z^*$. Take $f \in C^b$ such that $0 \leq \overline{f} \leq 1$, $\overline{f} = 1$ on W and $\overline{f} = 0$ on Z^*. Since $m(X - Z_d) = 0$ and since $f = 0$ on Z_d, we have that $\overline{m}(W) \leq \int_{\nu X} \overline{f} \, d\overline{m} = \overline{\phi}(\overline{f}) = \int_X f \, dm = 0$. Thus $\overline{m}(W) = 0$ for every zero set $W \subset \nu X - Z^*$. The regularity of $\overline{m}$ now implies $\overline{m}(\nu X - Z^*) = 0$ so that $G \subset Z^*$ as asserted.

We now claim that $\overline{g}_n \downarrow 0$ pointwise on G (which implies that $\overline{f}_i \downarrow 0$ on G.) If this is not so, then there is an $x_0 \in G$ and an $\alpha > 0$ such that $\overline{g}_n(x_0) \geq \alpha$ for all n. Define $Z = \bigcap_{n-1}^{\infty} \{x \in \nu X : \overline{g}_n(x) \geq \alpha\}$. Then $Z \cap Z^*$ is a zero set in νX with $x_0 \in Z \cap Z^*$. Since $x_0 \in \nu X$, we have that $X \cap Z \cap Z^* \neq \emptyset$. Let $x \in X$ $\cap Z \cap Z^*$. Since $Z_d = X \cap Z^*$, it follows that $x \in Z_d \cap Z^*$ which implies that $g_n(x) \geq \alpha > 0$ for all n. This contradicts the fact that $g_n \downarrow 0$ pointwise on Z_d. The proof is complete.

The following theorem summarizes the main facts.

Theorem 9.5. *Let X be a completely regular Hausdorff space (a D-space), and let M_{sc} (M_c) be endowed with the topology of uniform convergence on the pointwise bounded and equicontinuous sets in C. Then the following hold:*

1. *M_{sc} (M_c) is complete.*
2. *L is dense in M_{sc} (M_c).*
3. *The dual of M_{sc} (M_c) is C.*
4. *The restriction of the topology of M_{sc} (M_c) to X is the original topology of X.*

Proof. Both 1 and 2 are immediate from Theorem 9.4. That 3 holds is a consequence of Proposition 6.1 and the fact that the completion of a locally convex space has the same completion as the original space. Property 4 is a restatement of Proposition 6.2.

As a consequence of Theorem 9.5, we obtain the following result of Shirota. (See [4, p. 229].)

Theorem 9.6. *Let X be a D-space. Then X is realcompact if and only if X is complete for some uniform structure compatible with the topology of X.*

Proof. The points in νX may be identified with the ring homomorphisms of C into R. These are nonnegative so that νX is contained in M_c by Proposition 4.4. It is not hard to verify that νX is closed in M_c for the topology of uniform convergence on the sets in $\mathfrak{S}$. (In fact, νC is even closed in M_c for the topology $\sigma(M_c, C)$.) Hence if $X = \nu X$, then, by Theorem 9.5, X is complete for the uniformity induced on X from M_c. On the other hand, if X is complete for some

uniformity, then it is complete for the finest uniformity compatible with X [4, p. 231]. Since this is the uniformity generated by the family of all continuous pseudometrics on X, it is clear that it is the uniformity induced by e on X. Hence X is closed in M_c. The proof will be complete if we can show that X is dense in νX relative to the topology e on M_c.

In order to see that X is dense in νX, fix $x_0 \in \nu X$ and $B \in \mathfrak{E}$. It is enough to show that $X \cap (x_0 + B^\circ) \neq \emptyset$, or, what is equivalent, that $p_B(x - x_0) \leq 1$ for some $x \in X$ (where p_B is the Minkowski gauge of B°). Define $f(x) = p_B(x - x_0)$ for all $x \in \nu X$, and set $Z = \{x \in \nu X: f(x) \leq 1\}$. Then since $f \in C(\nu X)$, Z is a zero set in νX. This means that $Z \cap X \neq \emptyset$, and the proof is complete.

We can now prove the following analogue of Granirer's result (Corollary 8.8).

Corollary 9.7. *Let X be completely regular Hausdorff (a D-space).*

1. *If B is a pointwise bounded and equicontinuous subset of C, then B is relatively $\sigma(C, M_{sc})$-compact ($\sigma(C, M_c)$-compact).*

2. *If X is not a D-space, then there is a pointwise bounded equicontinuous set B in C which is not $\sigma(C, M_c)$-compact.*

Proof. 1. This is an immediate consequence of Theorem 9.5 and the Banach-Alaoglu theorem.

2. If X is not a D-space, then $M_{sc} \neq M_c$. Let M_c have the topology of uniform convergence on the sets in $\mathfrak{E}$. Then M_{sc} is a proper closed subspace of M_c by Theorem 9.5. By the Hahn-Banach theorem, there is a continuous linear functional on M_c which vanishes on X. Hence the dual of M_c is strictly larger than C. The result now follows by the Mackey-Arens theorem.

Remarks. 1. If X is compact (or pseudocompact), then Theorem 9.4 reduces to the Katětov-Pták theorem.

2. If X is discrete and nonmeasurable, then $M_c = L$.

Theorem 9.8. *Let X be a completely regular Hausdorff space and let M_c have the topology of uniform convergence on the pointwise bounded and equicontinuous subsets of C. Then the following are equivalent:*

1. *X is finite.*
2. *M_c is normable.*
3. *M_c is metrizable.*
4. *(L, e) is metrizable.*

Proof. $(1 \Rightarrow 2)$ This is obvious.

$(2 \Rightarrow 3)$ This is obvious.

$(3 \Rightarrow 4)$ This is obvious.

$(4 \Rightarrow 1)$ Take a sequence $\{B_n\} \subset \mathfrak{E}$ such that $B_n \subset B_{n+1}$ and such that

$\{B_n^{\circ}\}$ form a local base at zero for e. Also assume that $B_n^{\circ\circ} = B_n$ which is possible by Proposition 5.1 and the bipolar theorem. Then if $B \in \mathfrak{S}$, there is an n such that $B \subset B_n^{\circ\circ} = B_n$. Define $B_n^b = B_n \cap \{f \in C^b: \|f\|_X \leq n\}$. It is clear that if $B \in \mathfrak{S}^b$, then $B \subset B_n^b$ for some n. Hence $\{B_n^{b\circ}\}$ is a local base at zero for e^b. This means that e^b is metrizable. Hence (L, e^b) is a Mackey space. Hence by Theorem 8.9, X is discrete. If X is not finite, let $\{x_n\} \subset X$ be an infinite subset. For each n, take a number α_n such that $\sup\{f(x_n): f \in B_n\} < \alpha_n$. Finally, let f be any real-valued function on X such that $f(x_n) = \alpha_n$ for all n. Then $\{f\} \in \mathfrak{S}$ but $f \notin B_n$ for any n. This is a contradiction, and the proof is complete.

10. **Completion of L for the Mackey topology of (L, C).** Let m denote the Mackey topology on L for the pair (L, C), and let $\hat{L}^m$ denote the completion of this space. We will investigate the space $\hat{L}^m$ in this section. We will first develop some facts which will be very useful for this purpose. Proposition 10.2 seems to be of some interest in its own right.

Lemma 10.1. *Let B be a compact, convex and balanced subset of R^X with its product topology. If $\{f_n\}$ is a sequence of elements of B and if $\{a_n\}$ is a sequence of numbers with $\sum_{n=1}^{\infty} |a_n| \leq 1$, then the function $g = \sum_{n=1}^{\infty} a_n f_n$ belongs to B.*

Proof. Since B is pointwise bounded, for each $x \in X$, $\sup\{|f(x)|: x \in X\}$ is finite so that $g \in R^X$. Define $g_n = \sum_{k=1}^{n} a_k f_k$. Since $\sum^{\infty} |a_n| \leq 1$ and since B is convex and balanced, $g_n \in B$ for all $n \in N$. Furthermore, $g_n \to g$ pointwise. Since B is closed, it follows that $g \in B$. The proof is complete.

Proposition 10.2. *Let B be a compact convex and balanced subset of R^X with its product topology. If each function in B is uniformly bounded, then the family B is uniformly bounded. (That is, $\sup\{\|f\|_X: f \in B\} < \infty$.)*

Proof. Assume that the proposition is false, and let $\{f_n\}$ be a sequence in B with $\|f_n\|_X \uparrow +\infty$ as $n \to \infty$. For each $x \in X$ define a function $\beta_x: N \to R$ (where N is the set of positive integers) by $\beta_x(n) = f_n(x)$. Since B is pointwise bounded, $\{\beta_x: n \in N\} \subset l^{\infty}$.

Now let $\alpha = \{a_n\} \in l^1$ with $\|\alpha\|_1 \leq 1$, and define $f_\alpha = \sum_{n=1}^{\infty} a_n f_n$. Then $f_\alpha \in B$ by Lemma 10.1. Also, regarding l^{∞} as the Banach dual of l^1, we see that, for all $x \in X$, $|\beta_x(\alpha)| = |\sum_{n=1}^{\infty} a_n f_n(x)| = |f_\alpha(x)| \leq \|f_\alpha\|_X$. Thus it follows that $\{\beta_x: x \in X\}$ is a pointwise bounded family of continuous linear functionals on l^1. By the Banach-Steinhaus theorem, there is a number M such that $\|\beta_x\|_{\infty} \leq M$ for all $x \in X$. But this implies that $|f_n(x)| \leq M$ for all $x \in X$ and all $n \in N$. Hence $\|f_n\|_X \leq M$ for all $n \in N$ which contradicts the fact that $\|f_n\|_X \uparrow \infty$. The proof is complete.

Remark. It is easy to see that the hypothesis that B is convex and balanced cannot be omitted in the above theorem. Indeed take $X = N$ and $B = \{0\} \cup \{nf_n : n \in N\}$ where f_n is the characteristic function of $\{n\}$. Then B is compact in R^N and each function in B is uniformly bounded, although B itself is not uniformly bounded.

The following is a result of A. Grothendieck. A proof may be found in [17, p. 325].

Theorem 10.3. *Let X be a compact Hausdorff space and let $A \subset C(X)$. Then A is relatively weakly compact if and only if A is relatively $\sigma(C, L)$-compact and uniformly bounded.*

Proposition 10.4. (a) $\hat{L}^m$ *is a subspace of* M_{sc}.
(b) *If $m \in M$ and if m has compact support, then $m \in \hat{L}^m$.*

Proof. (a) Let $\phi \in \hat{L}^m$. Then by Proposition 5.1 and the completion theorem, ϕ is $\sigma(C, L)$-continuous on each $B \in \mathcal{E}$. Hence ϕ is in the completion of (L, e). It then follows by Theorem 9.4 that $\phi \in M_{sc}$.

(b) If m has compact support in X, then m is net-additive and, hence, a regular Borel measure by Proposition 1.8. Let G denote the support of m. Take any set $A \subset C$ which is $\sigma(C, L)$-compact, convex and balanced. By the completion theorem, we must show that if $\{f_i\} \subset A$ is any net with $f_i \rightarrow 0$ pointwise on X, then $\int_X f_i \, dm \rightarrow 0$.

Assume that there is a net $\{f_i\} \subset A$ such that $f_i \rightarrow 0$ pointwise on X while $|\int_X f_i \, dm| \geq \alpha > 0$ for all i. Now let A_G denote the set of functions formed by restricting the functions in A to G. Then A_G is a $\sigma(C(G), L(G))$-compact, convex and balanced subset of $C(G)$. Since G is compact, each function in A_G is uniformly bounded. Hence by Proposition 10.2 the family A_G is uniformly bounded. Hence by Theorem 10.3, A_G is a weakly compact subset of $C(G)$. Let f_i^* denote the restriction of f_i to G. Since A_G is weakly compact there is a subnet $\{g_j^*\}$ of $\{f_j^*\}$ such that $g_j^* \rightarrow 0$ weakly on G. Since m is supported on G, m belongs to the Banach dual of $C(G)$. Hence $\int_X g_j \, dm \rightarrow 0$. This contradicts the assumption that $|\int_X f_i \, dm| \geq \alpha > 0$ for all i. The proof is complete.

Theorem 10.5. *Let X be realcompact. Then the completion of L for the Mackey topology of the pair (L, C) is the space M_c with the Mackey topology of the pair (M_c, C).*

Proof. It is immediate from Proposition 10.4 that $\hat{L}^m = M_c$. Furthermore, the completion of a Mackey space is a Mackey space. (See [17, p. 262].) The proof is complete.

The following is a generalization of Grothendieck's result (Theorem 10.3).

Theorem 10.6. *Let X be realcompact and let $A \subset C$ be convex and balanced. Then A is relatively $\sigma(C, M_c)$-compact if and only if it is relatively $\sigma(C, L)$-compact.*

Proof. The completion of a locally convex space has the same equicontinuous sets in the dual space. (See [17, p. 261].) Hence if $A \subset C$ is $\sigma(C, L)$-compact, convex and balanced (and hence equicontinuous for the Mackey topology of the pair (L, C)), then A is equicontinuous for the Mackey topology for the pair (M_c, C) by Theorem 10.5. Hence $A^{\circ\circ}$ is $\sigma(C, M_c)$-compact by the Alaoglu theorem. Hence A is relatively $\sigma(C, M_c)$-compact. The proof is complete.

It seems reasonable to conjecture that for an arbitrary completely regular Hausdorff space, the completion of L for the Mackey topology of the pair (L, C) is the space M_{sc}. This conjecture is supported by Theorem 10.5. Furthermore, it has been proved by Pták [22, p. 187] that the conjecture holds if X is pseudocompact. The author hopes to return to this question in a future note. The following consequence of Theorem 10.5 is of some interest.

Theorem 10.7. *Let X be completely regular Hausdorff, and let M_c be endowed with the Mackey topology for the pair (M_c, C). Then M_c is complete and L is dense in M_c.*

Proof. For $f \in C$, let $\overline{f}$ denote its unique continuous extension to νX. Define a map $T: M_c(X) \to M_c(\nu X)$ by $T(m) = \overline{m}$ where $\overline{m}$ is the unique element in $M_c(\nu X)$ such that $\int_X f\, dm = \int_{\nu X} \overline{f}\, d\overline{m}$ for all $f \in C(X)$. It is easily verified that T is a topological isomorphism when $M_c(X)$ has the weak topology $\sigma(M_c(X), C(X))$ and $M_c(\nu X)$ has the weak topology $\sigma(M_c(\nu X), C(\nu X))$. Hence T is a topological isomorphism when $M_c(X)$ and $M_c(\nu X)$ have their respective Mackey topologies. Since $M_c(\nu X)$ is complete for its Mackey topology by Theorem 10.5 and since T^{-1} is a uniform space isomorphism, $M_c(X)$ is complete for its Mackey topology. It is clear that L is dense in M_c. The proof is complete.

The final question that arises is what is the completion (denoted by $\hat{L}^{bm}$) of L for the Mackey topology of the pair (L, C^b). We will present several results related to this question.

Proposition 10.8. (a) $\hat{L}^{bm} \subset M_s$.

(b) $\hat{L}^{bm}$ *contains the subspace of M consisting of those set functions which have compact support.*

Proof. (a) Since if $B \in \mathcal{E}^b$, then $B^{\circ\circ} \in \mathcal{E}^b$, this is immediate from Theorem 9.5 and the completion theorem.

(b) Since $\hat{L}^m \subset \hat{L}^{bm}$ by the completion theorem, this is immediate from Proposition 10.4.

The general representation of $\hat{L}^{bm}$ is unknown to the author at the present time, but he hopes to return to this question in the future. The following result is analogous to Theorem 10.7. It follows from the fact that C^b is a Banach space (see [26, Chapter VI, §1, Corollary 2 to Proposition 1]). We will include a proof for completeness.

Theorem 10.10. *Let X be completely regular Hausdorff, and let M be endowed with the Mackey topology for the pair (M, C^b). Then M is complete and L is dense in M.*

Proof. For $f \in C^b(X)$, let $\overline{f}$ denote its unique continuous extension to X. Define a map $T: M(X) \to M(\beta X)$ by $T(m) = \overline{m}$ where $\overline{m}$ is the unique element in $M(\beta X)$ such that $\int_X f \, dm = \int_X \overline{f} \, d\overline{m}$ for all $f \in C^b(X)$. Then T is a topological isomorphism for the weak topologies. Hence T is a topological isomorphism for the Mackey topologies. Since $M(\beta X) = M_c(\beta X)$, $M(\beta X)$ is complete for the Mackey topology by Theorem 10.5. Since T^{-1} is a uniform space isomorphism, it follows that $M(X)$ is complete for its Mackey topology. It is clear that L is dense in M. The proof is complete.

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